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A Two-Step Trigonometrically Fitted Semi-Implicit Hybrid Method for Solving Special Second Order Oscillatory Differential Equation

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ABSTRACT

In this paper, we derived a semi-implicit hybrid method (SIHM) which is a two-step method to solve special second order ordinary differential equations (ODEs). The SIHM which is three-stage and fourth-order is then trigonometrically fitted and denoted by TF-SIHM3(4). The method is constructed using trigonometrically fitted properties instead of using phase-lag and amplification properties. Numerical integration show that TF-SIHM3(4) is more accurate in term of accuracy compared to the existing explicit and implicit methods of the same order.

Keywords: Semi-Implicit Hybrid Method, Two-step Methods, Oscillatory problems, Trigonometrically-fitted.

1. Introduction

A lot of attention has been focused on the study of new methods for solving the initial value problem (IVP) for special second order ODE in the form of

$$y'' = f(\xi, y) \quad , y(\xi_0) = y_0 \quad , y'(\xi_0) = y'_0, \tag{1}$$

in which the solution is periodic or oscillating in nature. This type of ODE problems arise in the field of applied science such as satellite tracking, mechanics, quantum chemistry, molecular dynamic, electronics, astrophysics and so forth.

The popular methods that have been used to solve (1) numerically are Runge-Kutta (RK) method, multistep method, hybrid method, Runge-Kutta Nyström (RKN) method and many more. Many authors have developed and modified these methods by focusing on constructing methods with reduced dispersion and dissipation to enhance the accuracy of the methods. The analysis of dispersion error was first introduced by Bursa and Nigro (1980); dissipation (amplification)error is define as the distance of the computed solution from the cyclic solution and dispersion error (phase-lag) is the difference of the angle between the computed solution and the exact solution. D'Ambrosio et al. (2012) usedthe exponentially fitting technique in developing RK methods for solving ODE. Senu et al. (2014) have derived an explicit RK method with dispersion error of order infinity based on the method derived in Dormand (1996)for solving first order ODEs.

While for RKN methods, Van de Vyver (2007) has proposed a symplectic RKN method with minimize dispersion error. Many authors developed diagonally implicit RKN (DIRKN) methods with dispersion of higher order,such work can be seen in Van de Houwen et al. (1987), Senu et al. (2010), Senu et al. (2011), and Moo et al. (2014). In addition, by modifying certain coefficients ofthe existing RKN methods; some authors such as Papadopoulos et al. (2009) introduced a phase-fitted RKN method, while Kosti et al. (2012) developed optimized RKN method and Moo et al. (2013) also develop phase-fitted and amplification-fittedRKN methods. Zhang et al. (2013) developeda fifth-order trigonometrically fitted RKN method to solve radial Schrödinger equation and oscillatory problems. All the work mentioned above provedthat, having higher order of dispersion and dissipation improve the accuracy of a method.

On the other hand, Franco (1995) has proposed that (1) can be solved using a special multistep methods or explicit hybrid algorithms for solving second order ODEs. Later, Franco (2006) proposed explicit two-step hybrid methods up to order six for solving second order IVPs using the local truncation error and order condition developed by Coleman (2003). Several researcher such as Samat (2012), Fang and Wu (2008), Ahmad et al. (2013), and Senu et al. (2015), also work on developing and improving hybrid method using dispersion and dissipation properties for solving second order ODEs. Fang and Wu (2008), Ahmad et al. (2013), and Senu et al. (2015) have constructed a new kind of trigonometrically fitted hybrid method, zero-dissipative phase-fitted hybrid methods, and optimized hybrid methods respectively using the existing hybrid methods in Franco (2006). Other work on semi-implicit hybrid methods (SIHMs) with higher order of dispersion and dissipation relation can be seen in Ahmad et al. (2013) followed by Jikantoro et al. (2015) for solving oscillatory problems.

In order to use RK method to solve (1), the problem needs to be reduced to a system of first order ODEs. Therefore, it is more efficient if (1) can be solved directly using methods such as direct multistep method, RKN method and hybrid method. Hence in this paper, we are going to develop a new three-stage fourth-order SIHM then apply the trigonometrically fitting technique which is similar to the approach used byFang and Wu (2008)for solving oscillatory problems. The new method will be compared with several existing explicit and also implicit methods to prove that it is more efficient than the existing methods.

2. Derivation of New Trigonometrically Fitted Semi- Implicit Hybrid Method

2.1 Development of The New Semi-Implicit Hybrid Methods

The general formula of semi-implicit hybrid method for solving IVPs is given as

$$Y_i = (1 + c_i)y_n - c_i y_{n-1} + h^2 \sum_{i=1}^{s} a_{ij} f(\xi_n + c_j h, Y_j) \tag{2}$$

$$y_{n+1} = 2y_n - y_{n-1} + h^2 \sum_{i=1}^{s} b_i f(\xi_n + c_i h, Y_i) \tag{3}$$

where $i = 1, \cdots, s$, and $i \geq j$. The nodes are $c_1 = -1$,and $c_2 = 0$. We formulate equations (2) and (3) as below:

$$Y_1 = y_{n-1}, Y_2 = y_n, \tag{4}$$

$$Y_i = (1+c_i)y_n - c_i y_{n-1} + h^2 \sum_{i=1}^{s} a_{ij} f(\xi_n + c_j h, Y_j), \tag{5}$$

$$y_{n+1} = 2y_n - y_{n-1} + h^2 \left(b_1 f_{n_1} + b_2 f_n + \sum_{i=3}^{s} b_i f(\xi_n + c_i h, Y_i) \right) \tag{6}$$

where $i = 3, \cdots, s$, while functions $f_{n-1} = f(\xi_{n-1}, y_{n-1})$ and $f_n = f(\xi_n, y_n)$. The coefficients of b_i, c_i , and a_{ij} can be written in Table 1. The coefficients of the diagonal element (γ) are always equal for this method.

Table 1: s-stage semi-implicit hybrid methods

$$\begin{array}{c|cccccc}
-1 & 0 & & & & & \\
0 & 0 & 0 & & & & \\
c_3 & a_{3,1} & a_{3,2} & \gamma & & & \\
\vdots & \vdots & \vdots & \ddots & \ddots & & \\
c_s & a_{s,1} & a_{s,2} & \cdots & a_{s,s-1} & \gamma \\
\hline
 & b_1 & b_2 & \cdots & b_{s-1} & b_s
\end{array}$$

In this section, we derive a three stage fourth-order SIHM based on the order conditions, simplifying conditions and by minimizing of the error constant C_{p+1} of the method. The error constant is defined by

$$C_{\rho+1} = \|e_{\rho+1}(\xi_1), \cdots, e_{\rho+1}(\xi_k)\|_2 \tag{7}$$

where Coleman (2003) define that k is the number of order $\rho+2(\rho(\xi_i) = \rho+2)$, for the $\rho th-$ order method and $e_{\rho+1}(t_i)$ is the local truncation error. According to Coleman (2003), the conditions up to order five are listed as follows:

$$Order 2 \qquad \sum_{i=1}^{s} b_i = 1. \tag{8}$$

$$Order 3 \qquad \sum_{i=1}^{s} b_i c_i = 0. \tag{9}$$

$$\begin{aligned} &Order4 \quad && \sum_{i=1}^{s} b_i c_i^2 = \frac{1}{6}, \quad && \sum_{i=1}^{s} b_i c_i^2 = \frac{1}{12}. \qquad (10) \\ &Order5 \quad && \sum_{i=1}^{s} b_i c_i^2 = 0, \quad && \sum_{i=1}^{s} b_i c_i a_{ij} = \frac{1}{12}, \\ & && \sum_{i=1}^{s} b_i a_{ij} c_i = 0. && \qquad (11) \end{aligned}$$

where value of $i \geq j \geq k$. The methods also need to satisfy the simplifying condition for hybrid method which is:

$$\sum_{i}^{s} a_{ij} = \frac{c_i^2 + c_i}{2}, i = 3, \cdots, s. \qquad (12)$$

First, we use the algebraic order conditions up to order four (8)-(10), and also equation of simplifying condition (12) to derive the new method and then solved the equations simultaneously. We get the solution in term of free parameters a_{32}, a_{33}, and c_3 as follows:

$$a_{31} = -a_{32} - a_{33} + \frac{c_3}{2} + \frac{c_3^2}{2}, b_1 = \frac{1}{6(1+c_3)}, b_2 = \frac{6c_3 - 1}{6c_3}, b_3 = \frac{1}{6c_3(1+c_3)}$$

By assuming coefficient of $a_{32} = \frac{19}{24}$, $a_{33} = \frac{11}{600}$, and $c_3 = \frac{9}{10}$; we minimize the error constant from (7).

$$\|\tau^{(5)}\|_2 = 1.88398 \times 10^{(-2)}. \qquad (13)$$

where $\|\tau^{(5)}\|_2$ is the norms of the principal local truncation error coefficient for the fifth order method . Hence, we have the method semi-implicit hybrid method with three-stage fourth-order denoted as SIHM3(4) which is given in TABLE (2).

2.2 Adaption of Trigonometrically Fitted for Semi-Implicit Hybrid Method

To apply the trigonometrically fitted properties to SIHM, we consider SIHM3(4) in TABLE (2). The required internal stage (2) and the updated stage (3) are integrated exactly using the linear combination of the function $\sin(v\xi), \cos(v\xi)$

Table 2: The method SIHM3(4)

$$\begin{array}{c|ccc} -1 & 0 & & \\ 0 & 0 & 0 & \\ \frac{9}{10} & \frac{9}{200} & \frac{19}{24} & \frac{11}{600} \\ \hline & \frac{5}{57} & \frac{22}{27} & \frac{50}{513} \end{array}$$

for $v \in R$. Therefore, we get the following equations:

$$\begin{aligned} \cos(c_3\psi) &= 1 + c_3 - c_3\cos(\psi) \\ &\quad -\psi^2\left(a_{31}\cos(\psi) + a_{32} + a_{33}\cos(c_3\psi)\right) && (14) \\ \sin(c_3\psi) &= c_3\sin(\psi) + \psi^2(a_{31}\sin(\psi) - a_{33}\sin(c_3\psi)) && (15) \\ 2\cos(\psi) &= 2 - \psi^2(b_1\cos(\psi) + b_2 + b_3\cos(c_3\psi)) && (16) \\ b_1\sin(\psi) &= b_3\sin(c_3\psi) && (17) \end{aligned}$$

where $\psi = vh$ as v is fitted frequency and h is step size. By solving the equation (14) and (15) simultaneously with choice of coefficients $c_3 = \frac{9}{10}$ and $a_{32} = \frac{19}{24}$, we obtain a_{31} and a_{33} in term of ψ as below:

$$\begin{aligned} a_{31} &= -\frac{1}{120}\frac{E}{\psi^2 K}, \\ a_{33} &= -\frac{1}{60}\frac{F}{\psi^2} && (18) \end{aligned}$$

where

$$\begin{aligned} E &= 28311552\cos\left(\frac{\psi}{10}\right)^{18} - 120324096\cos\left(\frac{\psi}{10}\right)^{16} + 212336640\cos\left(\frac{\psi}{10}\right)^{14} \\ &\quad -201277440\cos\left(\frac{\psi}{10}\right)^{12} + 110702592\cos\left(\frac{\psi}{10}\right)^{10} - 35641344\cos\left(\frac{\psi}{10}\right)^{8} \\ &\quad +24320\cos\left(\frac{\psi}{10}\right)^{8}\psi^2 + 6488832\cos\left(\frac{\psi}{10}\right)^{6} - 42560\cos\left(\frac{\psi}{10}\right)^{6}\psi^2 \\ &\quad -624960\cos\left(\frac{\psi}{10}\right)^{4} + 22800\cos\left(\frac{\psi}{10}\right)^{4}\psi^2 - 3800\cos\left(\frac{\psi}{10}\right)^{2}\psi^2 \\ &\quad +28560\cos\left(\frac{\psi}{10}\right)^{2} + 95\psi^2 - 336, \end{aligned}$$

$$\begin{aligned}
F &= -60+15728640\cos\left(\frac{\psi}{10}\right)^{18}-66846720\cos\left(\frac{\psi}{10}\right)^{16}+117964800\cos\left(\frac{\psi}{10}\right)^{14}\\
&\quad -111820800\cos\left(\frac{\psi}{10}\right)^{12}+61501440\cos\left(\frac{\psi}{10}\right)^{10}-58368\cos\left(\frac{\psi}{10}\right)^{9}\\
&\quad +24320\cos\left(\frac{\psi}{10}\right)^{9}-19768320\cos\left(\frac{\psi}{10}\right)^{8}+116736\cos\left(\frac{\psi}{10}\right)^{7}\\
&\quad -48640\cos\left(\frac{\psi}{10}\right)^{7}\psi^2+3548160\cos\left(\frac{\psi}{10}\right)^{6}+31920\cos\left(\frac{\psi}{10}\right)^{5}\psi^2\\
&\quad -76608\cos\left(\frac{\psi}{10}\right)^{5}-316800\cos\left(\frac{\psi}{10}\right)^{4}+18240\cos\left(\frac{\psi}{10}\right)^{3}\\
&\quad -7600\cos\left(\frac{\psi}{10}\right)^{3}\psi^2+10800\cos\left(\frac{\psi}{10}\right)^{2}-1140\cos\left(\frac{\psi}{10}\right)+475\left(\frac{\psi}{10}\right)\psi^2,\\
K &= -1+262144\cos\left(\frac{\psi}{10}\right)^{18}-1114112\cos\left(\frac{\psi}{10}\right)^{16}+1966080\cos\left(\frac{\psi}{10}\right)^{14}\\
&\quad -1863680\cos\left(\frac{\psi}{10}\right)^{12}+1025024\cos\left(\frac{\psi}{10}\right)^{10}-329472\cos\left(\frac{\psi}{10}\right)^{8}\\
&\quad +59136\cos\left(\frac{\psi}{10}\right)^{6}-5280\cos\left(\frac{\psi}{10}\right)^{4}+180\cos\left(\frac{\psi}{10}\right)^{2}.
\end{aligned}$$

Then, we solve the linear system (16)-(17) with an additional order condition (8) for three-stage method which is

$$b_1+b_2+b_3=1 \tag{19}$$

to get b-values with choice of coefficient $c_3=\frac{9}{10}$. We get the value for b as below:

$$\begin{aligned}
b_1 &= -\frac{1}{2}\frac{Q}{M\psi^2},\\
b_2 &= \frac{1}{2}\frac{S}{N_1N_2\psi^2},\\
b_3 &= -\frac{T}{P\psi^2}
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
Q &= 16384\cos\left(\frac{\psi}{10}\right)^{14} - 8192\cos\left(\frac{\psi}{10}\right)^{13} - 53248\cos\left(\frac{\psi}{10}\right)^{12} + 24576\cos\left(\frac{\psi}{10}\right)^{11} \\
&+67584\cos\left(\frac{\psi}{10}\right)^{10} - 28160\cos\left(\frac{\psi}{10}\right)^{9} - 42240\cos\left(\frac{\psi}{10}\right)^{8} + 15360\cos\left(\frac{\psi}{10}\right)^{7} \\
&+13440\cos\left(\frac{\psi}{10}\right)^{6} - 4000\cos\left(\frac{\psi}{10}\right)^{5} + 16\cos\left(\frac{\psi}{10}\right)^{4}\psi^2 - 2064\cos\left(\frac{\psi}{10}\right)^{4} \\
&+432\cos\left(\frac{\psi}{10}\right)^{3} - 8\cos\left(\frac{\psi}{10}\right)^{3}\psi^2 + 148\cos\left(\frac{\psi}{10}\right)^{2} - 12\cos\left(\frac{\psi}{10}\right)^{2}\psi^2 \\
&-16\cos\left(\frac{\psi}{10}\right) + 4\cos\left(\frac{\psi}{10}\right)\psi^2 - 4 + \psi^2,
\end{aligned}$$

$$\begin{aligned}
S &= 1024\cos\left(\frac{\psi}{10}\right)^{10} + 512\cos\left(\frac{\psi}{10}\right)^{9} - 2560\cos\left(\frac{\psi}{10}\right)^{8} - 256\cos\left(\frac{\psi}{10}\right)^{8}\psi^2 \\
&-1024\cos\left(\frac{\psi}{10}\right)^{7}\psi^2 + 2240\cos\left(\frac{\psi}{10}\right)^{6} + 448\cos\left(\frac{\psi}{10}\right)^{6}\psi^2 + 672\cos\left(\frac{\psi}{10}\right)^{5} \\
&-800\cos\left(\frac{\psi}{10}\right)^{4} - 240\cos\left(\frac{\psi}{10}\right)^{4}\psi^2 - 160\cos\left(\frac{\psi}{10}\right)^{3}\psi^2 + 100\cos\left(\frac{\psi}{10}\right)^{2} \\
&+40\cos\left(\frac{\psi}{10}\right)^{2}\psi^2 + 10\cos\left(\frac{\psi}{10}\right)\psi^2 - 4 - \psi^2 \\
T &= 16384\cos\left(\frac{\psi}{10}\right)^{14} - 61440\cos\left(\frac{\psi}{10}\right)^{12} + 92160\cos\left(\frac{\psi}{10}\right)^{10} - 70400\cos\left(\frac{\psi}{10}\right)^{8} \\
&+28800\cos\left(\frac{\psi}{10}\right)^{6} + 16\cos\left(\frac{\psi}{10}\right)^{4}\psi^2 - 6064\cos\left(\frac{\psi}{10}\right)^{4} + 580\cos\left(\frac{\psi}{10}\right)^{2} \\
&-20\cos\left(\frac{\psi}{10}\right)^{2}\psi^2 - 20 + 5\psi^2 \\
N_1 &= 16\cos\left(\frac{\psi}{10}\right)^{5} - 8\cos\left(\frac{\psi}{10}\right)^{4} - 20\cos\left(\frac{\psi}{10}\right)^{3} + 8\cos\left(\frac{\psi}{10}\right)^{2} + 5\cos\left(\frac{\psi}{10}\right) - 1, \\
N_2 &= 1 - 12\cos\left(\frac{\psi}{10}\right)^{2} + 16\cos\left(\frac{\psi}{10}\right)^{4}, \\
M &= 8192\cos\left(\frac{\psi}{10}\right)^{14} - 4096\cos\left(\frac{\psi}{10}\right)^{13} - 26624\cos\left(\frac{\psi}{10}\right)^{12} + 12288\cos\left(\frac{\psi}{10}\right)^{11}
\end{aligned}$$

$$
\begin{aligned}
& +33792\cos\left(\frac{\psi}{10}\right)^{10} - 14080\cos\left(\frac{\psi}{10}\right)^{9} - 21120\cos\left(\frac{\psi}{10}\right)^{8} + 7680\cos\left(\frac{\psi}{10}\right)^{7} \\
& +6720\cos\left(\frac{\psi}{10}\right)^{6} - 2016\cos\left(\frac{\psi}{10}\right)^{5} - 1016\cos\left(\frac{\psi}{10}\right)^{4} + 228\cos\left(\frac{\psi}{10}\right)^{3} \\
& +62\cos\left(\frac{\psi}{10}\right)^{2} - 9\cos\left(\frac{\psi}{10}\right) - 1, \\
P \;=\; & \cos\left(\frac{\psi}{10}\right)^{14} - 28672\cos\left(\frac{\psi}{10}\right)^{12} + 39424\cos\left(\frac{\psi}{10}\right)^{10} - 26880\cos\left(\frac{\psi}{10}\right)^{8} \\
& +9408\cos\left(\frac{\psi}{10}\right)^{6} - 16\cos\left(\frac{\psi}{10}\right)^{5} - 1568\cos\left(\frac{\psi}{10}\right)^{4} + 20\cos\left(\frac{\psi}{10}\right)^{3} \\
& +98\cos\left(\frac{\psi}{10}\right)^{2} - 5\cos\left(\frac{\psi}{10}\right) - 1.
\end{aligned}
$$

We transform the above formulae into Taylor series expansions as below:

$$
\begin{aligned}
a_{31} &= \frac{9}{200} - \frac{189}{12500}\psi^2 - \frac{19321}{2500000}\psi^4 - \frac{33394877}{11250000000}\psi^6 + O(\psi^8), && (21)\\
a_{32} &= \frac{11}{600} - \frac{7917}{400000}\psi^2 - \frac{658851}{80000000}\psi^4 - \frac{60824490931}{20160000000000}\psi^6 \\
&\quad +O(\psi^8), && (22)\\
b_1 &= \frac{5}{57} + \frac{23}{4560}\psi^2 + \frac{158653}{5774560000}\psi^4 + \frac{2003803}{114912000000}\psi^6 + O(\psi^8), && (23)\\
b_2 &= \frac{22}{27} - \frac{49}{6480}\psi^2 - \frac{72973}{272160000}\psi^4 - \frac{1383449}{163296000000}\psi^6 + O(\psi^8), && (24)\\
b_3 &= \frac{50}{513} + \frac{31}{12312}\psi^2 - \frac{4139}{517104000}\psi^4 - \frac{556343}{62052480000}\psi^6 + O(\psi^8). && (25)
\end{aligned}
$$

The values of a_{31}, a_{32}, b_1, b_2, and b_3 are constants for constant v and h. This new method is denoted as trigonometrically fitted three-stage fourth-order semi-implicit hybrid method (TF-SIHM3(4)).

3. Problems Tested and Numerical Integrations

The new method is tested using set of linear and nonlinear test problems in literature. Methods are tested for large interval $[0, 10000]$ to indicate that the new TF-SIHM3(4) is suitable for integrating oscillatory problems. We evaluate the efficiency using absolute error which is defined by

$$\text{Absolute error} = max|y(\xi_n) - y_n|$$

where $y(\xi_n)$ is the exact solution and y_n is the computed solution. The problems are listed as below:

PROBLEM 1 (Inhomogeneous system in Lambert and Watson (1976))

$$\begin{aligned}
\frac{d^2 y_1(\xi)}{d\xi^2} &= -v^2 y_1(\xi) + v^2 f(\xi) + f''(\xi), \\
y_1(0) &= a + f(0), \quad y_1'(0) = f'(0), \\
\frac{d^2 y_2(\xi)}{d\xi^2} &= -v^2 y_2(\xi) + v^2 f(\xi) + f''(\xi), \\
y_2(0) &= f(0), \qquad y_2'(0) = va + f'(0),
\end{aligned}$$

Analytical solution is given as $y_1(\xi) = a\cos(v\xi) + f(\xi)$ and $y_2(\xi) = a\sin(v\xi) + f(\xi)$. The value of $f(\xi)$ is equal to $e^{(-0.05\xi)}$, $v = 20$ and $a = 0.1$.

PROBLEM 2 (Inhomogeneous system in Franco (2006))

$$y''(\xi) = \begin{pmatrix} \frac{101}{2} & -\frac{99}{2} \\ -\frac{99}{2} & \frac{101}{2} \end{pmatrix} y(\xi) = \delta \begin{pmatrix} \frac{93}{2}\cos(2\xi) & -\frac{92}{2}\sin(2\xi) \\ \frac{93}{2}\sin(2\xi) & -\frac{92}{2}\cos(2\xi) \end{pmatrix}$$

$$y(0) = \begin{pmatrix} -1+\delta \\ 1 \end{pmatrix}, \quad y'(0) = \begin{pmatrix} -10 \\ 10+2\delta \end{pmatrix}, \quad \delta = 10^{-3}.$$

The Eigen-value of the problem are $v = 10$ and $v = 2$. The fitted frequency is choose to be $v = 10$ because it is dominant than $v = 2$. The analytical solution is given by

$$y(\xi) = \begin{pmatrix} -\cos(10\xi) - \sin(10\xi) + \delta\cos(2\xi) \\ \cos(10\xi) + \sin(10\xi) + \delta\cos(2\xi) \end{pmatrix}$$

PROBLEM 3 (Homogeneous studied in Chakravarti and Worland (1971))

$$y''(\xi) = -y(\xi), \quad y(0) = 0, \quad y'(0) = 1.$$

The exact solution is $y(\xi) = \sin(\xi)$ and the fitted frequency is $v = 1$.

PROBLEM 4 (Inhomogeneous equation studied in Papadopoulos et al. (2009))

$$y''(\xi) = -v^2 y(\xi) + (v^2 - 1)\sin(\xi), \quad y(0) = 1, \quad y'(0) = v + 1.$$

The fitted frequency is $v = 10$. The analytical solution is $y(\xi) = \cos(v\xi) + \sin(v\xi) + \sin(\xi)$.

PROBLEM 5 (Two-Body problem studied in Papadopoulos et al. (2009))

$$y_1''(\xi) = \frac{-y_1(\xi)}{r^3}, \quad y_2''(\xi) = \frac{-y_2(\xi)}{r^3} \tag{26}$$

where $r\sqrt{y_1^2 + y_2^2} =$, $y_1(0) = 1$, $y_2(0) = 0$, $y_1'(0) = 1$, $y_2'(0) = 0$, and fitted frequency, $v = 1$. The analytical solutions are $y_1(\xi) = \cos(\xi)$ and $y_2(\xi) = \sin(\xi)$.

The notations that are used in Figure 1-5:

1. TF-SIHM3(4):Trigonometrically-fitted three-stage fourth-order Semi-implicit hybrid method developed in this paper.
2. SIHM3(4):Three-stage fourth-order SIHM developed in this paper.
3. E-HM3(4):Explicit three-stage fourth-order hybrid method derived by Franco (2006).
4. RKN4:Explicit three-stage fourth-order RKN by Hairer et al. (2010).
5. DIRKN3(4):Three-stage fourth- order DIRKN in Senu et al. (2010).
6. PFRKN4(4):Phase-fitted explicit four-stage fourth-order RKN by Papadopoulos et al. (2009).
7. DIRKN(HS):Three-stage fourth-order DIRKN derived in Sommeijer (1987).

In analyzing the numerical results, the logarithm of the maximum global error are plotted against the CPU time taken in second for all the methods. From Figures 1-5, we observed that TF-SIHM3(4) is the most efficient

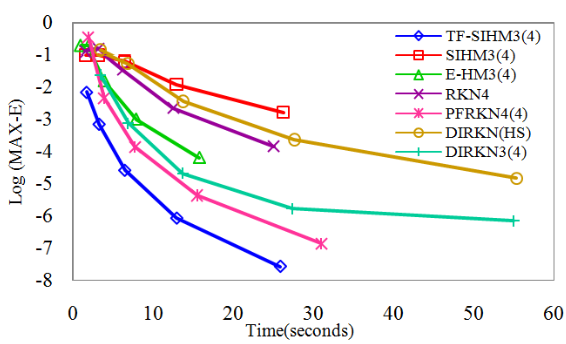

Figure 1: The efficiency curves for problem 1 with $h = \frac{0.125}{2^i}$, for $i = 2, \ldots, 6$

method for integrating second order oscillatory ODEs, followed by PFRKN4(4), DIRKN3(4), E-HM3(4), RKN4, DIRKN(HS), and SIHM3(4). This show that Trigonometrically-fitting the method improve the accuracy of the original method,

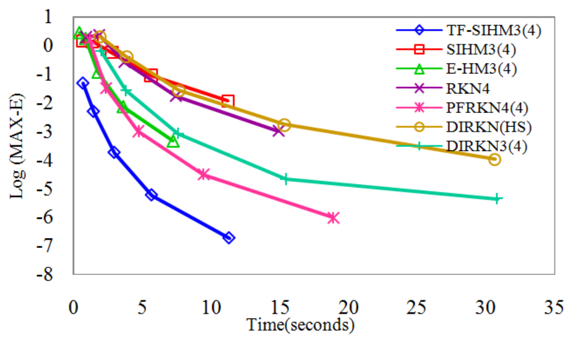

Figure 2: The efficiency curves for problem 2 with $h = \frac{0.125}{2^i}$, for $i = 1, \ldots, 5$

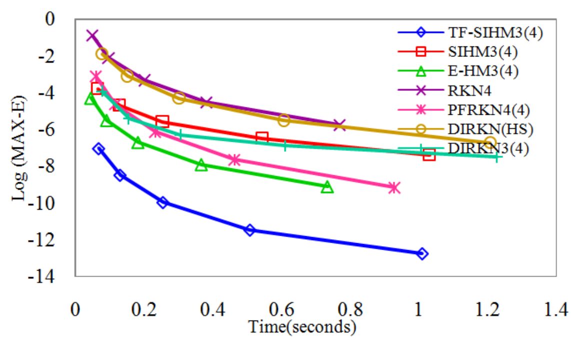

Figure 3: The efficiency curves for problem 3 with $h = \frac{0.125}{2^i}$, for $i = 2, \ldots, 6$

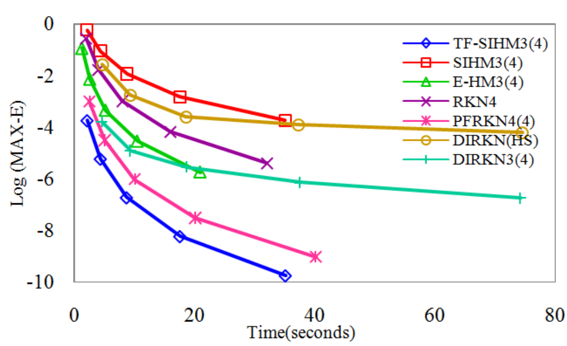

Figure 4: The efficiency curves for problem 4 with $h = \frac{0.125}{2^i}$, for $i = 3, \ldots, 7$

SIHM3(4). Even though the new TF-SIHM3(4) is a semi- implicit method and fairly expensive in terms of computational time, the method is noticeably better in accuracy compared to other explicit and implicit methods .

4. Conclusion

In this research,atrigonometrically fitted three-stage fourth-order semi-implicit hybrid method denoted as TF-SIHM3(4) is constructed and presented. The new developedmethod is suitable for solving either linear or nonlinear oscillatory problems. From the result shown in Figures1-5, we can conclude that TF-SIHM3(4) is very efficient in term of accuracy compared to other well-known existing implicit and explicit methods of the same order in the scientific literature.

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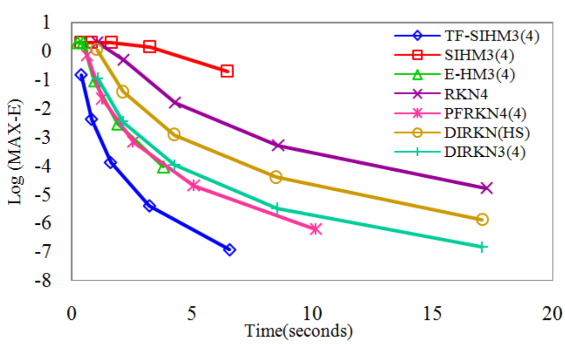

Figure 5: The efficiency curves for problem 5 with $h = \frac{0.125}{2^i}$, for $i = 0, \ldots, 4$

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